Journal of Statistical Physics, Vol. 75, Nos. 5/6, 1994

Nonlocality of the Misra–Prigogine–Courbage Semigroup

Z. Suchanecki,[1,2] I. Antoniou,[1,3] and S. Tasaki[1]

Received March 29, 1993; final January 24, 1994

We show that the Markov semigroups constructed by Misra, Prigogine, and Courbage through nonunitary similarity transformations of Kolmogorov systems are not implementable by local point transformations, i.e., they are not the Frobenius–Perron semigroups associated with noninvertible point transformations, in contrast with the semigroups obtained by coarse-graining projections. Our result is a straightforward generalization of the proof of the nonlocality of the similarity transformation given by Goldstein, Misra, and Courbage and also of the previous illustration by Misra and Prigogine for the baker transformation and completes the characterization of the Misra–Prigogine–Courbage semigroups.

KEY WORDS: Dynamical systems; Kolmogorov systems; Markov semigroups; irreversibility; intertwining transformations.

1. INTRODUCTION

The problem of irreversibility in statistical physics lies in understanding the relation between reversible dynamical laws and the observed entropy-increasing evolutions. The prototypes of such irreversible evolutions are Markov processes such as kinetic or diffusive processes. The nonunitary transformation theory of irreversibility, originated by Prigogine *et al.*[1,2] and carried forward by Prigogine, Misra, Courbage, and others,[2-13] poses the question in the following way: what types of unitary groups U_t can be

[1] International Solvay Institutes for Physics and Chemistry, 1050 Brussels, Belgium.

[2] Hugo Steinhaus Center and Institute of Mathematics, Wroclaw Technical University, Wroclaw, Poland.

[3] Theoretische Natuurkunde, Free University of Brussels, 1050 Brussels, Belgium. ULBG028@bearn.bitnet.

0022-4715/94/0600-0919$07.00/0

intertwined with Markov semigroups M_t, $t \geqslant 0$, through nonunitary transformations Λ:

$$M_t \Lambda = \Lambda U_t, \qquad t \geqslant 0 \tag{1}$$

Here U_t is the unitary evolution of square-integrable densities associated with the dynamics S_t, $t \in (-\infty, +\infty)$ or $t = 0, \pm 1, \pm 2,...$, on the phase space Y equipped with a σ-algebra $\mathscr{B}$ and an invariant measure μ which describes equilibrium:

$$U_t \rho(y) = \rho(S_t^{-1} y) \tag{2}$$

The irreversible Markov semigroup M_t, $t \geqslant 0$, on the space $\mathscr{L}^2(Y, \mathscr{B}, \mu)$ has the following properties:

1. M_t are contractions,

$$\|M_t \rho\|^2 \leqslant \|\rho\|^2 \tag{3}$$

2. M_t preserve probabilities, i.e.,

$$M_t \rho \geqslant 0 \qquad \text{if} \quad \rho \geqslant 0 \tag{4}$$

$$\int_Y d\mu \, M_t \rho = \int_Y d\mu \, \rho \tag{5}$$

The last condition means that

$$M_t^\dagger 1 = 1 \tag{6}$$

Properties (6) and (3) imply on $\mathscr{L}^2(Y, \mathscr{B}, \mu)$ that[13]

$$M_t 1 = 1 \tag{7}$$

3. Irreversible approach to equilibrium is described by the condition

$$\|M_t \rho - 1\|^2 \to 0 \qquad \text{as} \quad t \to \infty \tag{8}$$

for every square-integrable density ρ.

Misra, Prigogine, and Courbage showed in fact that the unitary evolution U_t of highly unstable dynamical systems, such as Kolmogorov systems defined in Section 2, can be intertwined (1) either by a similarity[4-8,9,11,12]

$$M_t = \Lambda U_t \Lambda^{-1} \tag{9}$$

or by a coarse-graining projection $\Lambda = P$,[3,9,10,12]

$$M_t = P U_t P \tag{10}$$

with irreversible Markov semigroups. The nonunitary intertwining transformation Λ effectively incorporates the objective limitations to predictability due to the dynamical instability.

As the action of Λ is highly delocalizing,[6] such a transformation cannot be the multiplication operator by a function on the phase space, nor can it be the induced operator from an underlying point transformation of the phase space. One would therefore expect the resulting Markov semigroup M_t will also not be implementable by point transformations. However, in the case of coarse-graining projections, Antoniou and Gustafson[13] showed that the resulting irreversible Markov semigroup (10) is implementable by a noninvertible point transformation, i.e., $M_t \geqslant 0$, is the exact Markov semigroup associated with a noninvertible point transformation $\tilde{S}_t$, $t \geqslant 0$, acting on the coarse-grained phase space $\tilde{Y}$ with respect to the K-partition (see Section 2) of the original phase space Y:

$$M_t^\dagger \rho(\tilde{y}) = \rho(\tilde{S}_t \tilde{y}), \qquad t \geqslant 0 \tag{11}$$

We use the above expression (11) for the adjoint Koopman semigroups[14] because the transformations $\tilde{S}_t$ are not invertible. Therefore the corresponding expression (2) should be modified to the Frobenius–Perron operators formula,[14] which takes into account the inverse branches of $\tilde{S}_t$.

We remark in passing that the implementability of the Misra–Prigogine–Courbage semigroup M_t associated with the coarse-graining projection P not only shows that it is possible to intertwine reversible dynamics with irreversible dynamics, but was also used[13] to prove a converse to the Misra–Prigogine–Courbage result, namely that irreversible Markov semigroups arising from projections can be dilated to reversible dynamics. The positive dilation construction of ref. 13 is based upon Rokhlin's[15,14] natural extension of exact systems to Kolmogorov systems.

In view of the result in ref. 13 it is necessary to know also whether the semigroup (9) associated with similarities is implementable, i.e., if it is also the Frobenius–Perron semigroup associated with a noninvertible point transformation. Misra and Prigogine[7] have shown that for a specific model, the baker system, with a specific choice of Λ transformation the resulting semigroup is nonlocal, i.e., nonimplementable. The purpose of this paper is to generalize this result. We show that the semigroup (9), contrary to the semigroup (10), is nonlocal for all Kolmogorov systems and for all choices of Λ transformations. The idea of the proof is essentially the same as for the nonlocality of the Λ transformation as given by Goldstein *et al.*[6] However, we relax the condition of strict monotonicity for the function $\Lambda(\tau)$ while retaining the condition $\Lambda(\tau) < \Lambda(\infty)$.

2. THE MISRA–PRIGOGINE–COURBAGE CONSTRUCTION

Before we present the Misra–Prigogine–Courbage similarity transformation we recall the definition of Kolmogorov systems. Kolmogorov systems[16,14] are highly unstable dynamical systems S_t, t real or integer, characterized by a measurable partition ξ (K-partition) of the phase space Y which evolves asymmetrically, namely:

(i) ξ is progressively refined:

$$S_t\xi \leqslant \leqslant S_{t'}\xi, \qquad t < t' \tag{12a}$$

(ii) ξ approaches the finest point partition ν in the far future

$$\bigvee_{t \geqslant 0} S_t\xi = \nu \tag{12b}$$

(iii) ξ approaches the coarsest one-cell partition ε in the far past

$$\bigwedge_{t \leqslant 0} S_t\xi = \varepsilon \tag{12c}$$

The conditional expectations P_t over the cells of the time-evolved K-partition $S_t\xi$, t real or integer, define a family of projections which inherit the properties of the K-partition:

$$\text{(i)} \quad P_t \leqslant P_{t'}, \qquad t < t' \tag{13a}$$

$$\text{(ii)} \quad \operatorname*{s-lim}_{t \to \infty} P_t = I \tag{13b}$$

$$\text{(iii)} \quad \operatorname*{s-lim}_{t \to -\infty} P_t = P_e \tag{13c}$$

I is the identity operator on $\mathscr{L}^2(Y, \mathscr{B}, \mu)$ and P_e is the orthoprojection onto the one-dimensional space of constants which includes the equilibrium density,

$$P_e\rho = \int_Y d\mu\, \rho$$

As the K-partition becomes more and more refined in the future, the observation of the system through the K-partition becomes more and more operationally unattainable. This limitation is expressed through the Λ transformation, which is defined on the space $\mathscr{L}^2(Y, \mathscr{B}, \mu)$:

$$\Lambda\rho = \int_{-\infty}^{+\infty} \Lambda(\tau)\, dP_\tau\, \rho + P_e\rho \qquad \text{for} \quad \text{K-flows} \tag{14a}$$

$$\Lambda\rho = \sum_{-\infty}^{+\infty} \Lambda(\tau)(P_\tau - P_{\tau-1})\rho + P_e\rho \qquad \text{for} \quad \text{K-cascades} \tag{14b}$$

$\Lambda(\tau)$ is any positive function on the reals or integers with the following properties:

(a) Λ is decreasing with $\Lambda(\infty)=0$ and $\Lambda(\tau)<\Lambda(-\infty)\leqslant 1$, for all $t\in\mathbf{R}$. Here we relax the original assumption of Misra, Prigogine, and Courbage that Λ is strictly decreasing

(b) Λ is a logarithmically concave function.

The properties (a) and (b) imply that the function Λ has the form

$$\Lambda(\tau)=e^{-\phi(\tau)} \tag{15}$$

where $\phi(\tau)$ is a positive convex function which increases to $+\infty$ as $\tau\to\infty$. A specific choice for Λ is the function

$$\Lambda(\tau)=\frac{1}{1+e^{\tau}} \tag{16}$$

The transformation (16) was used by Misra and Prigogine[7] in their study of the baker transformation. A further restriction on the possible forms of $\Lambda(\tau)$ was proposed by Suchanecki[12] on the basis of Gnedenko's theorem.

The transformation Λ leads to the irreversible Markov semigroup (9) on the space $\mathscr{L}^2(Y,\mathscr{B},\mu)$,

$$M_t=\Lambda U_t\Lambda^{-1}=\int_{-\infty}^{+\infty}\frac{\Lambda(\tau)}{\Lambda(\tau-t)}\,dP_\tau\,U_t+P_eU_t \tag{17a}$$

for flows, $t\geqslant 0$,

$$M_t=\Lambda U_t\Lambda^{-1}=\sum_{-\infty}^{+\infty}\frac{\Lambda(\tau)}{\Lambda(\tau-t)}\,(P_\tau-P_{\tau-1})\,U_t+P_eU_t \tag{17b}$$

for cascades, t positive integer.

The irreversible semigroup properties of M_t are guaranteed because the function $\Lambda(\tau)/\Lambda(\tau-t)$ is a bounded, decreasing function of τ for every $t>0$.[4–6] A simple proof of this fact goes as follows: It is enough to show that the function $\tau\mapsto\phi(\tau-t)-\phi(\tau)$ is decreasing. Let $\tau_1<\tau_2$; then $0<t/(\tau_2+t-\tau_1)<1$. Since the function ϕ is convex, we have

$$\begin{aligned}\phi(\tau_1-t)+\phi(\tau_2)&=\frac{t}{\tau_2+t-\tau_1}\,\phi(\tau_1-t)+\frac{\tau_2-\tau_1}{\tau_2+t-\tau_1}\,\phi(\tau_2)\\&\quad+\frac{\tau_2-\tau_1}{\tau_2+t-\tau_1}\,\phi(\tau_1-t)+\frac{t}{\tau_2+t-\tau_1}\,\phi(\tau_2)\\&\geqslant\phi\left(\frac{t(\tau_1-t)+(\tau_2-\tau_1)\tau_2}{\tau_2+t-\tau_1}\right)+\phi\left(\frac{(\tau_2-\tau_1)(\tau_1-t)+t\tau_2}{\tau_2+t-\tau_1}\right)\\&=\phi(\tau_2-t)+\phi(\tau_1)\end{aligned}$$

which implies the desired result

$$\phi(\tau_1 - t) - \phi(\tau_1) \geqslant \phi(\tau_2 - t) - \phi(\tau_2)$$

We remark before closing this section that the semigroup M_t has the following property, which we shall need in the next section.

Lemma 1. The operator

$$M_t - \int_{-\infty}^{+\infty} P_\tau U_t \, dg(\tau), \qquad t \geqslant 0 \tag{18}$$

with

$$g(\tau) \equiv -\frac{\Lambda(\tau)}{\Lambda(\tau - t)} \tag{19}$$

vanishes or is positivity-preserving on $\mathscr{L}^2(Y, \mathscr{B}, \mu)$.

Proof. Using integration by parts, we can write M_t as the following Lebesgue–Stieltjes integral (see Lemma in ref. 9, p. 75)):

$$\begin{aligned} M_t &= \int_{-\infty}^{\infty} P_\tau U_t \, d\left(-\frac{\Lambda(\tau)}{\Lambda(\tau - t)}\right) + \lim_{\tau \to +\infty} \frac{\Lambda(\tau)}{\Lambda(\tau - t)} U_t \\ &\quad - \lim_{\tau \to -\infty} \frac{\Lambda(\tau)}{\Lambda(\tau - t)} P_e U_t + P_e U_t \\ &= \int_{-\infty}^{\infty} P_\tau U_t \, d\left(-\frac{\Lambda(\tau)}{\Lambda(\tau - t)}\right) + \lim_{\tau \to +\infty} \frac{\Lambda(\tau)}{\Lambda(\tau - t)} U_t \end{aligned}$$

where we have used

$$\lim_{\tau \to -\infty} \frac{\Lambda(\tau)}{\Lambda(\tau - t)} = 1$$

Since the function $\Lambda(\tau)/\Lambda(\tau - t)$ is positive and decreasing, the limit

$$\lim_{\tau \to \infty} \frac{\Lambda(\tau)}{\Lambda(\tau - t)}$$

also exists and it is nonnegative and (18) follows immediately. Goldstein *et al.*[16] proved essentially the same result under the stronger assumption that the function $\Lambda(\tau)/\Lambda(\tau - t)$ is strictly monotonically decreasing.

3. NONIMPLEMENTABILITY OF THE SEMIGROUP $\Lambda U_t \Lambda^{-1}$

The result is expressed as the following:

Theorem. The semigroup $M_t = \Lambda U_t \Lambda^{-1}$, $t \geqslant 0$, is not implementable, i.e., there does not exist a measurable point transformation $\tilde{S}_t$ of Y such that $\tilde{S}_t$ preserves μ and for which

$$M_t^\dagger \rho(y) = \rho(\tilde{S}_t y) \qquad \text{for all} \quad t \geqslant 0 \tag{20}$$

Since we deal with the adjoint operator $M_t^\dagger$, it is convenient to use the following simple lemma.

Lemma 2. Let M be a linear operator on $\mathscr{L}^2$ which is implementable, (20), by a measure-preserving point transformation $\tilde{S}$. Then for each measurable set Δ such that its image $\tilde{S}(\Delta)$ under $\tilde{S}$ is also measurable, the following holds:

$$\int_{Y - \tilde{S}(\Delta)} d\mu \, M \mathbf{1}_\Delta = 0 \tag{21}$$

Proof of Lemma 2. For any measurable set Δ for which $\tilde{S}\Delta$ is also measurable, we have

$$\int_{Y - \tilde{S}(\Delta)} d\mu \, M \mathbf{1}_\Delta = \int_Y (M \mathbf{1}_\Delta) \cdot \mathbf{1}_{Y - \tilde{S}(\Delta)} \, d\mu(y) = (M \mathbf{1}_\Delta \mid \mathbf{1}_{Y - \tilde{S}(\Delta)})_{L^2}$$

$$= (\mathbf{1}_\Delta \mid M^\dagger \mathbf{1}_{Y - \tilde{S}(\Delta)})_{L^2} = \int_\Delta \mathbf{1}_{Y - \tilde{S}(\Delta)}(\tilde{S}(y)) \, d\mu(y)$$

$$= \int_{\tilde{S}(\Delta)} \mathbf{1}_{Y - \tilde{S}(\Delta)}(y) \, d\mu(y) = 0 \quad \text{QED}$$

Proof of the Theorem. Now, suppose M_t is implementable by the measure-preserving transformation $\tilde{S}_t$ on Y, (20). Consider a measurable set Δ such that $\tilde{S}_t(\Delta)$ is also measurable and $0 < \mu(\Delta) < 1$. Thus by Lemma 2, we have

$$\int_{Y - \tilde{S}_t(\Delta)} d\mu \, M_t \mathbf{1}_\Delta = 0 \tag{22}$$

However, similar to ref. 6, we shall show that

$$\int_B d\mu \, M_t \mathbf{1}_\Delta > 0 \qquad \text{for each measurable } B \text{ with } \mu(B) > 0 \tag{23}$$

which contradicts Lemma 2.

Indeed, using Lemma 1 in Section 2 and applying Fubini's theorem, we have

$$\int_B d\mu\, M_t \mathbf{1}_\Delta \geqslant \int_B d\mu \int_{-\infty}^{\infty} P_\tau U_t \mathbf{1}_\Delta\, dg(\tau) = \int_{-\infty}^{\infty} \left(\int_B P_\tau U_t \mathbf{1}_\Delta\, d\mu \right) dg(\tau) \tag{24}$$

but $U_t \mathbf{1}_\Delta = \mathbf{1}_{S_t(\Delta)}$ and

$$P_\tau \mathbf{1}_{S_t(\Delta)} \to P_e \mathbf{1}_{S_t(\Delta)} = \mu(S_t(\Delta)) = \mu(\Delta) \qquad \text{when} \quad \tau \to -\infty$$

Therefore

$$\int_B d\mu\, P_\tau U_t \mathbf{1}_\Delta \to \mu(\Delta)\, \mu(B) \qquad \text{when} \quad \tau \to -\infty \tag{25}$$

Consequently there is a number τ_0 such that for $\tau < \tau_0$

$$\int_B d\mu\, P_\tau U_t \mathbf{1}_\Delta > \tfrac{1}{2}\mu(\Delta)\, \mu(B)$$

Then, using (24) and (25), we obtain

$$\int_B d\mu\, M_t \mathbf{1}_\Delta \geqslant \int_{-\infty}^{\tau_0} \left(\int_B d\mu\, P_\tau U_t \mathbf{1}_\Delta \right) dg(\tau)$$
$$> \tfrac{1}{2}\mu(\Delta)\, \mu(B)(g(\tau_0) - g(-\infty)) > 0$$

where the inequality $g(\tau_0) - g(-\infty) > 0$ follows from the fact that $\Lambda(\tau) < \Lambda(-\infty)$.

The proof for cascades follows if we extend the function $\Lambda(\tau)$, $\tau = 0, \pm 1, \pm 2,...$, to the whole real line in such a way that, for $\tau < x < \tau + 1$, the value $\Lambda(x)$ is equal to the value on the segment joining the points $(\tau, \Lambda(\tau))$ and $(\tau + 1, \Lambda(\tau + 1))$.

The proof follows also from the result of Goldstein *et al.*[6] The point is that the argument that if $\Lambda(\tau)$ is strictly monotonically decreasing, then the transformation Λ is positivity improving, given on p. 121 of ref. 6, also proves that the same conclusion follows, without "strictly," provided $\Lambda(\tau) < \Lambda(-\infty)$. This is because the integrand on the right-hand side of the key inequality on line 12 of p. 121 of ref. 6 has a positive limit as $\tau \to -\infty$, as stated on line 17. It follows immediately that the RHS is positive, since every interval $(-\infty, \tau)$ is now assigned positive measure, namely $\Lambda(-\infty) - \Lambda(\tau)$. Moreover, the conditions assumed on $\Lambda(\tau)$, corresponding to (a) and (b) in this paper, guarantee that $\tilde{\Lambda}_t(\tau) = \Lambda(\tau + t)/\Lambda(\tau)$ on p. 122 of ref. 6 satisfies $\tilde{\Lambda}_t(\tau) < 1 = \tilde{\Lambda}_t(-\infty)$, as well as being decreasing, so that the semigroup is in fact nonimplementable.

However, because of the importance of the result also in connection with recent developments,[13] we presented a suitably streamlined presentation of the nonlocality of the Misra–Prigogine–Courbage semigroup.

4. CONCLUDING REMARKS

1. Our results together with the result in ref. 13 completes the characterization of the semigroups constructed by Misra, Prigogine, and Courbage in their theory of irreversibility. Namely the semigroups obtained through coarse-grained projections (10) are locally implementable by point transformations,[13] while the semigroups obtained by nonunitary intertwining transformations (9) are not locally implementable.

2. As the semigroup (9) arising from similarities is nonimplementable by point transformations, it is not possible to construct unitary dilations through the natural extension[15,14] as in the case of the semigroups (10) arising projections.[13] However, one can always construct[21] more general dilations of irreversible Markov semigroups which can also be applied to the nonimplementable semigroup (9).

3. The space $\mathscr{L}^2$ in the formulation and discussion of the problem can be replaced by any space $\mathscr{L}^p$, $1 \leqslant p < \infty$ (see ref. 12 for the construction of M_t in this case).

4. We have shown the nonlocality of the Misra–Prigogine–Courbage semigroup which provides a time-asymmetric representation of the unstable dynamical systems. Our result shows that the specific illustration of Misra and Prigogine[7] is true in general. The nonunitary similarities Λ provide nonlocal representations of Kolmogorov systems. This result is in conformity with our recent result[17] that the baker transformations admit nonlocal spectral decompositions in extended functional spaces without nonunitary similarities. Nonlocality is therefore a typical property of intrinsically irreversible systems.[2,18–20]

ACKNOWLEDGMENTS

We are grateful to Profs. I. Prigogine, B. Misra, K. Gustafson and R. Goodrich for several fruitful discussions and useful comments. Also we are grateful to a referee for comments and suggestions for a proof based directly on ref. 6. This work is partly supported by the Belgian Government (under the contract "Pole d'attraction interuniversitaire"), the European Communities Commission (contract 27155.1/BAS), the U. S. Department of Energy, grant FG05-88ER13897, and the Robert A. Welch Foundation.

REFERENCES

1. I. Prigogine, C. George, F. Henin, and L. Rosenfeld, *Chem. Scripta* **4**:5–32 (1973).
2. I. Prigogine, *From Being to Becoming* (Freeman, San Francisco, 1980).
3. B. Misra and I. Prigogine, *Prog. Theor. Phys. Suppl.* **69**:101–110 (1980).
4. B. Misra, I. Prigogine, and M. Courbage, *Physica A* **98**:1–26 (1979).
5. M. Courbage and B. Misra, *Physica A* **104**:359–377 (1980).
6. S. Goldstein, B. Misra, and M. Courbage, *J. Stat. Phys.* **25**:111–126 (1981).
7. B. Misra and I. Prigogine, *Lett. Math. Phys.* 7:421–429 (1983).
8. M. Courbage, *Physica A* **122**:359–377 (1983).
9. B. Misra and I. Prigogine, Time probability and dynamics, in *Long Time Predictions in Dynamical Systems*, C. Horton, L. Reichl, V. Szebehely, eds. (Wiley, New York, 1983), pp. 21–43.
10. R. Goodrich, K. Gustafson, and B. Misra, *J. Stat. Phys.* **43**:317–320 (1986).
11. Z. Suchanecki and A. Weron, *Expositiones Math.* **8**:67–79 (1990).
12. Z. Suchanecki, *Physica A* **187**:249–266 (1992).
13. I. Antoniou and K. Gustafson, *Physica A* **197**:153–166 (1993).
14. A. Lasota and M. C. Mackey, *Probabilistic Properties of Deterministic Systems* (Cambridge University Press, Cambridge, 1985).
15. V. Rokhlin, *Am. Math. Soc. Transl.* **39**:1–36 (1964).
16. I. Cornfeld, S. Fomin, and Ya. Sinai, *Ergodic Theory* (Springer, Berlin, 1982).
17. I. Antoniou and S. Tasaki, *Physica A* **190**:303–329 (1992).
18. I. Antoniou and I. Prigogine, Intrinsic irreversibility in classical and quantum mechanics, in *The Concept of Probability*, E. Bitsakis and C. Nicolaides, eds. (Kluwer, Dordrecht, 1989), pp. 223–234.
19. I. Prigogine, *Phys. Rep.* **219**:109–120 (1992).
20. I. Antoniou and I. Prigogine, *Physica A* **192**:43–464 (1993).
21. I. Antoniou, K. Gustafson, Z. Suchanecki, and A. Vershik, Dilation of Markov processes to dynamical systems, to appear.